Rational maps of curves

Let C be a curve, i.e. a variety with $\dim C = 1$.
If C is not in the plane, we can't use calculus methods to study its singularities. Instead, we just rely on the local rings.

Def: $P \in C$ is a <u>simple</u> or <u>smooth</u> point if $\mathcal{O}_P(C)$ is a DVR. (This agrees w/ our def for plane curves.)
In this case, let ord_P^C or just ord_P be the order function on $k(C)$ defined by $\mathcal{O}_P(C)$.

(Recall: $\mathrm{ord}_P(z) = \{ n \mid z = \underset{\text{unit}}{u}\, \underset{\text{uniformizing parameter}}{t}^{n} \}$)

More generally,
let K be a field containing k.

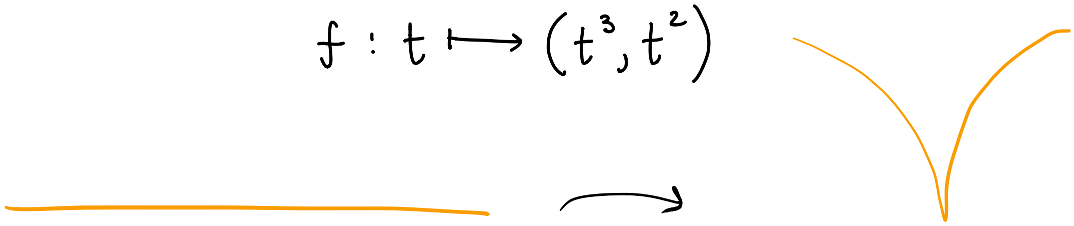
Def: A is a <u>local ring of K</u> if $A \subseteq K$ is a local subring, $k \subseteq A$, and K is the field of fractions of A. If A is also a DVR, it's called a <u>DVR of K</u>.

e.g. if V is a variety, $P \in V$, then $\mathcal{O}_P(V)$ is a local ring of $k(V)$. The converse doesn't hold!

For example, if C is a curve, and R is a local ring of $k(C)$, R might not be $\mathcal{O}_P(C)$ for some $P \in C$:

EX: Consider again the map from $\mathbb{A}^1$ to $V = V(x^2 - y^3)$ defined

$$f : t \longmapsto (t^3, t^2)$$



(This is just the dehomogenization of the example in the previous section.)

This induces a map $f^* : k[x,y]/(x^2 - y^3) \longrightarrow k[t]$ defined

$$x \longmapsto t^3$$
$$y \longmapsto t^2$$

Since this is injective (f is dominant), it extends to a map $f^* : k(V) \longrightarrow k(\mathbb{A}^1) = k(t)$.

Since this is a map of fields, it's injective, and in fact, it's surjective, and thus an isomorphism, since

$$\frac{x}{y} \longmapsto \frac{t^3}{t^2} = t.$$

(Gaurav's) Question: What happens to the non-DVR $\mathcal{O}_{(0,0)}(V)$ in $k(\mathbb{A}^1)$ since $\mathbb{A}^1$ only has smooth points?

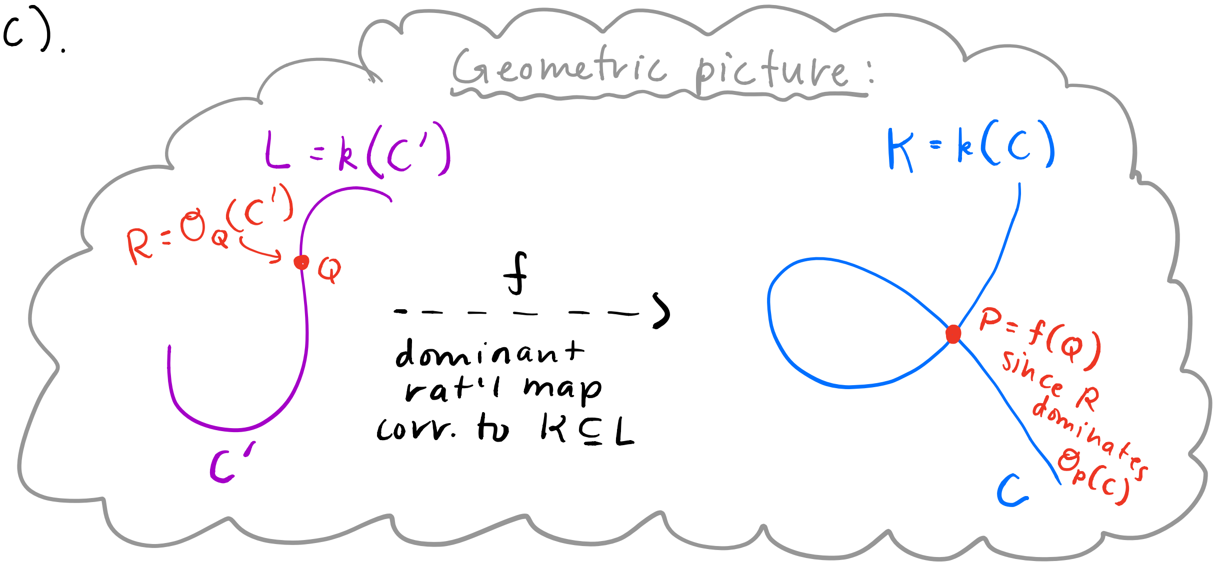
$\mathcal{O}_{(0,0)}(V) = \left\{ \frac{f}{g} \,\middle|\, g(0,0) \neq 0 \right\}$ has image $\left\{ \frac{f(t^3, t^2)}{g(t^3, t^2)} \,\middle|\, g \notin (t) \right\}$

which is thus a local ring w/ max'l ideal (t^2, t^3), which is not principal since t is not in the ring.

Thus, this is a local ring that doesn't correspond to a point on $\mathbb{A}^1$! However, it is dominated by $\mathcal{O}_0(\mathbb{A}^1)$, which means $0 \mapsto (0,0)$.

In fact, for every DVR of $k(C)$, there is a point $P \in C$ whose local ring it dominates. More generally:

Theorem: Let C be a projective curve, $K = k(C)$. Let L be a field containing K, R a DVR of L s.t. $R \not\supset K$. Then there is a unique point $P \in C$ s.t. R dominates $\mathcal{O}_P(C)$.



Pf: Uniqueness: Suppose $P \neq Q$ and R dominates both $\mathcal{O}_P(C)$ and $\mathcal{O}_Q(C)$.

Find an open affine U containing both P and Q. Then $\exists$ $f \in \Gamma(U) \subseteq k(C)$ s.t. $f(P) = 0$ but $f(Q) \neq 0$.

Thus, $f \in m_P \subseteq$ max ideal of R, but $\frac{1}{f} \in \mathcal{O}_Q(C) \subseteq R$, so

ord $f > 0$ but ord $\frac{1}{f} \geq 0$, a contradiction.

Existence: Suppose $C \subseteq \mathbb{P}^n$, closed. We can assume $C \cap U_i \neq \emptyset$ for all $i = 1, \ldots, n+1$. (Otherwise choose a smaller n)

Then $\Gamma_h(C) = k[x_1, \ldots, x_{n+1}]/I_p(C) = k[a_1, \ldots, a_{n+1}]$ where a_i is the image of x_i, and $a_i \neq 0$.

Then $\frac{a_i}{a_j} \in k(C) = K \subseteq L$. Let $N = \max_{i,j} \text{ord}(a_i/a_j)$.

Assume $\text{ord}(a_j/a_{n+1}) = N$ for some j (possibly after a change of coordinates). Then for all i, we have

$$\text{ord}(a_i/a_{n+1}) = \text{ord}((a_j/a_{n+1})(a_i/a_j)) = N - \text{ord}(a_j/a_i) \geq 0.$$

If $C' = C \cap U_{n+1}$, then $\Gamma(C') = k[a_1/a_{n+1}, \ldots, a_n/a_{n+1}]$, so

$$\Gamma(C') \subset R.$$

Let $m \subseteq R$ be the maximal ideal, and $J = m \cap \Gamma(C')$. Then J is prime, so $W := V(J) \subseteq C'$.

If $W = C'$, then $J = 0$ so every nonzero elt of $\Gamma(C')$ is a unit in R. Thus, $K \subset R$, which contradicts our assumption.

Thus $W = \{P\}$, a point, and $m_P = (J) \subseteq m$, so R dominates $\mathcal{O}_P(C') = \mathcal{O}_P(C)$. $\square$

From this, we get a beautiful geometric corollary, hinted at above:

Corollary: Let C' be a curve and C a projective curve. Let $f: C' \dashrightarrow C$ be a rational map. Then every simple point is in the domain of f. In particular, if C' is nonsingular, f is a morphism.

Pf: Note that f is either constant or dominant: the closure of the image is either C or finitely many points. If it is finitely many points $P_1, \ldots, P_r$, $r > 1$, then $f^{-1}(P_i)$ is closed, but not C', so it is finitely many points. Thus, the domain is finitely many pts, which is not open, a contradiction.

If f is constant, it is defined on all of C'.

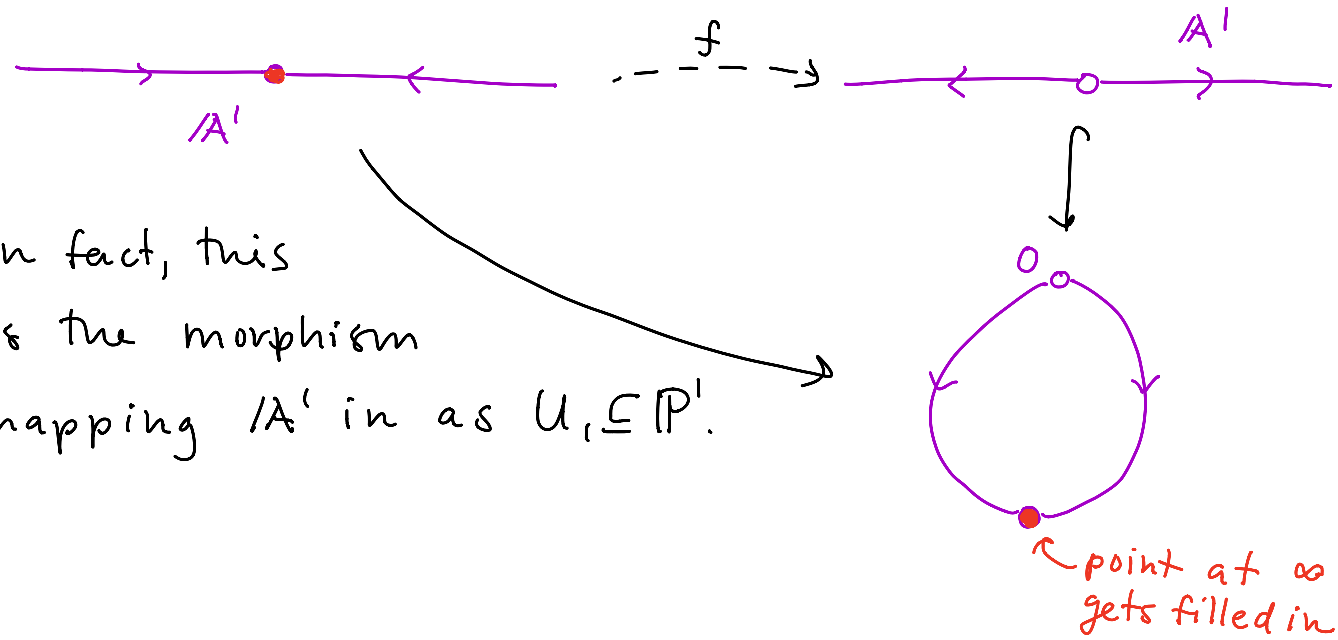
So assume f is dominant. Then $f^*: K(C) \hookrightarrow K(C')$. Let $P \in C'$ be simple, $R = \mathcal{O}_P(C')$. Then to apply the theorem, we just need to show $K(C) \not\subseteq R$.

Suppose $K(C) \subseteq R \subseteq K(C')$. $K(C')$ is a finite algebraic extension of $K(C)$ (they're both alg. function fields in one variable over R), so R must be a field (exer), but R is a DVR, which is a contradiction. $\square$

(Roughly, if there is a missing smooth point in the domain, then the image of a sequence "converging" to that point will also converge since projective varieties are "compact" in the Euclidean topology.)

Ex: Recall the rational map $f: \mathbb{A}^1 \dashrightarrow \mathbb{A}^1$ defined $x \mapsto 1/x$. This can't be extended to all of $\mathbb{A}^1$. However, if we make the target $\mathbb{P}^1$, w/ $\mathbb{A}^1$ sitting inside it as U_2, we get

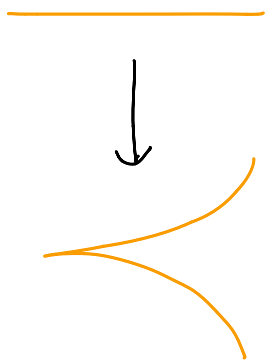
$$x \longmapsto [1/x : 1] = [1 : x],$$ which is a morphism!



In fact, this is the morphism mapping $\mathbb{A}^1$ in as $U_1 \subseteq \mathbb{P}^1$.

Note that we can't always extend maps to singular points even when it's possible to AS A FUNCTION:

Ex: Consider the map from $\mathbb{P}^1 \to V(x^3 - y^2 z) \subseteq \mathbb{P}^2$ defined $[s:t] \mapsto [s^2 t : s^3 : t^3]$



The map is 1-to-1, and an isomorphism away from 0, but the inverse function is not a morphism (for one, multiplicity at a point cannot decrease by a HW prob).

The next two corollaries follow immediately:

Cor: C projective curve, C' nonsingular, then there's a 1-to-1 correspondence between dominant morphisms $f: C' \to C$ and homomorphisms $f^*: k(C) \to k(C')$.

Cor: Two nonsingular projective curves are isomorphic $\Longleftrightarrow$ their fields of rational functions are isomorphic.

Note: This is not true for higher dim. varieties! $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ have the same field of rat'l functions, but they're not isomorphic: Not every pair of curves in $\mathbb{P}^1 \times \mathbb{P}^1$ intersects.

Lastly, we know that the points on a nonsingular curve C correspond to DVRs of $k(C)$. In fact, the converse holds:

Theorem: If C is a nonsingular projective curve, there is a one-to-one correspondence between the points of C and DVRs of $k(C)$, where $P \in C$ corresponds to $\mathcal{O}_P(C) \subsetneq k(C)$.

Pf: We just need to show that any DVR of $k(C)$ is a local ring at some point $P \in C$.

Let R be a DVR of $k(C)$. Then by the previous theorem, R dominates a unique $\mathcal{O}_P(C)$. Let $(a) = m_P \subset \underset{\text{max'l ideal of } R}{m}$.

Suppose $R \neq \mathcal{O}_P(C)$. Let $z \in R \setminus \mathcal{O}_P(C)$. Then $\mathrm{ord}_P(z) < 0$, so $\frac{1}{z} \in (a) \Rightarrow a$ is a unit in R, a contradiction. Thus, $R = \mathcal{O}_P(C)$. □

This raises the following question:

Q: If C is a singular curve, do the DVRs of $k(C)$ determine a smooth curve birational to C?

In the next section, we discuss a method of systematically "resolving" singularities in order to find a smooth birational model of a curve.